Boundary observability of time discrete Schrödinger equations

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Abstract: In this paper we study the boundary observability estimate of time discrete Schrödinger equations in a bounded domain. By means of a time discrete version of the classical multiplier technique, we prove the uniform observability inequality of the solutions in an appropriate filtered space in which the high frequency components have been filtered. In this way, the well-known boundary observability property of the Schödinger equation can be reproduced as the limit, as $h \to 0$, of the observability of the time discrete one. Better than the existing result in Ervedoza et al. (2008), our alterative proof shows the rigorous relationship between the filtering parameter and the optimal observation time T. Moreover, the latter one tends to zero as the time scale tends to zero. Finally, the optimality of the order of the filtering parameter is also established for lower dimensional case.

Keywords: Schrödinger equation; boundary observability; time discretisation; multipliers; convergence.

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1 Introduction

Let Ω be an open bounded set of $\mathbb{R}^d$ with boundary $\Gamma = \partial\Omega$ of class C^3. We consider a partition (Γ_0, Γ_1) of Γ given by

$$\Gamma_0 = \Gamma_{x^0} = \{x \in \Gamma : m(x) \cdot \nu(x) > 0\}, \tag{1}$$

$$\Gamma_1 = \{x \in \Gamma : m(x) \cdot \nu(x) \leq 0\}, \tag{2}$$

where x^0 is a fixed point of $\mathbb{R}^d, m(x) = x - x^0$, and $\nu(x)$ is the unit normal vector to Γ at $x \in \Gamma$ pointing towards the exterior of Ω, and '$\cdot$' denotes the scalar product in $\mathbb{R}^d$.

Let us consider the following homogeneous Schrödinger equation

$$\begin{cases} i\varphi_t + \Delta\varphi = 0 & \text{in } Q = \Omega \times (0,T) \\ \varphi = 0 & \text{on } \Sigma = \Gamma \times (0,T) \\ \varphi(0) = \varphi_0 & \text{in } \Omega. \end{cases} \tag{3}$$

Here $\varphi = \varphi(x,t)$ is the state and is a complex valued function. In Machtyngier (1994), it is shown that for arbitrary time interval $(0,T)$, system (3) is exact observable on the boundary Γ_0. More precisely, for any $T > 0$, there exists a constant $C = C(T) > 0$ such that for any initial data $\varphi_0 \in H_0^1(\Omega)$, the following inequality holds for the solution of (3):

$$\| \varphi \|_{H_0^1(\Omega)}^2 \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma. \tag{4}$$

The above observability estimate has a wide range of applications on controllability, stabilisation, inverse problem, etc. There is also an intensive literature providing observability results implicitly for Schrödinger equations by various methods including microlocal analysis (Bardos et al., 1992; Lebeau, 1992), multipliers (Machtyngier, 1994), Carleman estimates (Baudouin and Puel, 2002; Lasiecka and Triggiani, 1992), etc.

Note that there is another class of conditions on $(\Omega\Gamma_0)$ guaranteeing (4), which is so-called *geometric control condition* (GCC, for short) introduced in Bardos et al. (1992). It asserts that all rays of geometric optics in Ω intersect the subset of the boundary Γ_0 in a uniform time 1. Indeed, this is the case when one introduces the microlocal analysis technique (Bardos et al., 1992; Lebeau, 1992).

Our goal in this paper is to develop a theory allowing to get results for time discrete systems as a direct consequence of those corresponding to the time-continuous ones. Especially, we focus on the multiplier conditions of the boundary, i.e., Γ_0 satisfies (1).

Let us first present a natural discretisation of continuous system (3). For any $h > 0$, we denote by φ_k the approximation of the solution φ of system (3) at time $t_k = kh$ for any $k = 0, \ldots, K$ with $K = T/h$. We consider the following *implicit midpoint* time discretisation of system (3):

$$\begin{cases} i\dfrac{\varphi_{k+1} - \varphi_k}{h} + \Delta\left(\dfrac{\varphi_{k+1} + \varphi_k}{2}\right) = 0, & \text{in } \Omega,\ k = 0, \ldots, K-1 \\ \varphi_k = 0, & \text{on } \Gamma,\ k = 0, \ldots, K \\ \varphi_0 \text{ is given,} & \text{in } \Omega. \end{cases} \tag{5}$$

Note that (5) is a discrete version of (3).

As we will show in Lemma 6.1, the conservation law for the time discrete equation (3) holds, i.e., $\|\varphi_k\|^2_{L^2(\Omega)} = \|\varphi_0\|^2_{L^2(\Omega)}$ and $\|\varphi_k\|^2_{H_0^1(\Omega)} = \|\varphi_0\|^2_{H_0^1(\Omega)}$ for any $k = 0,...,K.$ Consequently the scheme under consideration is stable and its convergence (in the classical sense of numerical analysis) is guaranteed in an appropriate functional setting.

The uniform exact observability problem for system (5) is formulate similarly to the continuous one: *To find a positive constant* $C > 0,$ *independent of* h, *such that the solution* φ_k *of system* (5) *satisfy*

$$\|\varphi_0\|^2_{H_0^1(\Omega)} \leq Ch \sum_{k=0}^{K-1} \int_{\Gamma_0} \left| \frac{\partial}{\partial \nu} \left(\frac{\varphi_{k+1} + \varphi_k}{2} \right) \right|^2 d\sigma. \tag{6}$$

for all initial data in an appropriate class.

Clearly, (6) is a discrete version of (4). Accordingly, system (5) is said to be observable if there is a constant C such that (6) holds.

The first result of this paper is of negative nature. Indeed, as we shall see in Theorem 2.1, the observability inequality fails for system (5) if the initial data are taken in $H_0^1(\Omega)$. Note that, from the proof of Theorem 2.1 below, one will see that this negative result is related to the fact that the number of time-steps is finite; while the space in which the solutions involve is infinite dimensional. Of course, one cannot expect to observe infinite number of information by means of finite number of observations. Note that the similar result holds for time discrete wave equation and the methodology of the proof there is the same as in our model (see Zhang et al., 2009 for more details).

Accordingly, to obtain the uniform observability property (6) one needs to restrict the solutions of (5) by filtering the high frequency components. The filtering method has been applied successfully in the context of observability of time discrete conservative linear systems (Ervedoza et al., 2008), controllability of time discrete heat equations (Zheng, 2008) and space semi-discrete schemes for wave equations (Infante and Zuazua, 1999; Zuazua, 1999, 2005). Indeed, the subject of observation of the time discrete Schrödinger equation under consideration is roughly stated in Ervedoza et al. (2008) as an application of an abstract model. However, due to the limitation of the techniques they applied, a very cursory result is stated and show the existence of some time $T > 0,$ with which (6) holds with suitable filtered initial data (Th. 4.2 of Ervedoza et al., 2008). The result is rough, without any detailed discussion on the optimality of the observable time T and the order of the filtering parameter. In this paper, we not only develop a direct proof for solving this uniform observability problem by means of a discrete version of the classical multiplier approach, but also obtain a sharp observation time T, i.e., (6) holds for any $T > 0.$ It can be seen not as a complementarity but an improvement of the previous results in Ervedoza et al. (2008).

The rest of the paper is organised as follows. In Section 2, we present the lack of the observability of system (6) without filtering. Section 3 is devoted to establish a fundamental identity by means of multipliers, which will play an important role in the sequel. The uniform observability result for (6) is presented in Section 4. In Section 5 we show the optimality of the filtering parameter in the uniform observability result. Finally we state some technical Lemmas as the complements of the previous proofs.

2 Lack of observability

This section is devoted to prove the following negative observability result:

Theorem 2.1: For any given $h > 0$ and any nonempty open subset Γ_0 of Γ, system (5) is not observable with $\varphi_0 \in H_0^1(\Omega)$.

Proof: We emphasise that, in this proof, h is fixed so that the system under consideration involves only a finite number of time-steps while it is a distributed parameter system (infinite-dimensional one) in space. This is precisely the main reason for the lack of observability property.

Let μ_j^2 and Φ_j are eigenvalues and eigenfunctions of the Laplacian with Dirichlet boundary condition, i.e.,

$$\begin{cases} -\Delta\Phi_j = \mu_j^2\Phi_j, \text{ in } \Omega \\ \Phi_j = 0, \quad \text{on } \Gamma. \end{cases} \text{Put } f^n = \sum_{j=1}^n \left|\mu_j\right|^{-d/2-1_{\Phi_j}}. \tag{7}$$

(Recall that d is the dimensions of Ω). By Weyl's formula (Guillemin, 1979), $\mu_k \sim C(\Omega)k^{1/d}$ as $k \to \infty$. Therefore,

$$\| f^n \|_{H_0^1(\Omega)}^2 = \sum_{j=1}^{n} \left|\mu_j\right|^{-d} \to \infty, \quad \text{as } n \to \infty; \tag{8}$$

while, $\{f^n\}_{n\geq 1}$ is bounded in $H^{-s}(\Omega)$ for all $s > -1$.

It is obvious that $f^n \in H^2(\Omega) \cap H_0^1(\Omega)$ for any n. We choose the initial data of (5) to be $\varphi_0^n = f^n$ and denote the corresponding solution by $\{\varphi_k^n\}_{k=0}^K$. Note that $\varphi_1^n, \ldots, \varphi_K^n$ are inductively determined by the following iterative elliptic systems

$$i\varphi_{k+1}^n + \frac{h}{2}\Delta\varphi_{k+1}^n = i\varphi_k^n - \frac{h}{2}\Delta\varphi_k^n, \quad k = 0, \ldots K-1. \tag{9}$$

By standard elliptic regularity theory, it is easy to see that $\varphi_k^n \in H^2(\Omega) \cap H_0^1(\Omega)$ for any $n \in \mathbb{N}$.

One can also rewrite (9) as

$$\varphi_{k+1}^n + \varphi_k^n = \frac{2i}{h}(-\Delta)^{-1}(\varphi_{k+1}^n - \varphi_k^n), \quad k = 0, \ldots K-1.$$

Using the standard elliptic regularity theory, for any $\tau \leq 2$, it holds

$$\sum_{k=1}^{K-1} \left\|\varphi_{k+1}^n + \varphi_k^n\right\|_{H^\tau(\Omega)} \leq \sum_{k=1}^{K-1} C(h)\left\|\varphi_{k+1}^n + \varphi_k^n\right\|_{H^{\tau-2}(\Omega)} \leq C(h)\left\|f^n\right\|_{H^{\tau-2}(\Omega)}. \tag{10}$$

In the second inequality we use the fact that $\| \varphi_k^n \|_{H^{\tau-2}(\Omega)} = \| f^n \|_{H^{\tau-2}(\Omega)}$ for any k. Hence, for any given $h > 0$ and $3/2 < \tau < 2$, using trace theorem, it follows from (10) that

$$h\sum_{k=1}^{K-1}\int_\Gamma\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}^n+\varphi_k^n}{2}\right)\right|^2 d\Gamma \le C(h)\sum_{k=1}^{K-1}\|\varphi_{k+1}^n+\varphi_k^n\|_{H^\tau(\Omega)}^2 \le C(h)\|f^n\|_{H^{\tau-2}(\Omega)}^2. \tag{11}$$

Now, recalling that $\{f^n\}_{n\ge1}$ is bounded in $H^{-s}(\Omega)$ for all $s>-1$ and $\varphi_0^n=f^n$, taking (8) and (11) into account, we obtain that

$$\lim_{n\to\infty}\frac{\|f^n\|_{H_0^1(\Omega)}^2}{h\sum_{k=1}^{K-1}\int_\Gamma\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}^n+\varphi_k^n}{2}\right)\right|^2 d\Gamma}=\infty. \tag{12}$$

Thus, the observability inequality fails. Consequently, system (5) is not observable (even when $\Gamma_0=\Gamma$).

3 Identity via multipliers

In this section, we will establish an identity for the solution of the system (5) by means of the discrete multiplier techniques. As we shall see, it plays the crucial role in the proof of Theorem 4.1.

We have the following Lemma:

Lemma 3.1: We denote by $q_k=q(x,t_k)$ where $q(x,t)\in C^2(\overline{Q},\mathbb{R}^d)$. For every solution of (5) with $\varphi_0\in\mathcal{D}(Q)$, the corresponding identity holds:

$$\begin{aligned}
&\frac{1}{2}h\sum_{k=0}^{K-1}\int_\Gamma\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 q_k\cdot\nu d\sigma\\
&=\frac{1}{2}\operatorname{Im}\int_\Omega(\varphi_K q_K\cdot\nabla\overline{\varphi}_K-\varphi_0 q_0\cdot\nabla\overline{\varphi}_0)dx\\
&\quad+\frac{1}{2}\operatorname{Im}h\sum_{k=0}^{K-1}\int_\Omega\frac{q_{k+1}-q_k}{h}\cdot\nabla\varphi_k\overline{\varphi}_k dx\\
&\quad+\frac{1}{2}\operatorname{Re}h\sum_{k=0}^{K-1}\int_\Omega\nabla\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\cdot\left(\nabla\operatorname{div}_x q_k\right)\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)dx\\
&\quad+\operatorname{Re}h\sum_{k=0}^{K-1}\int_\Omega\sum_{i,j}\left(\frac{\partial q_{j,k}}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)\frac{\partial}{\partial x_i}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right)dx.
\end{aligned}\tag{13}$$

Remark 3.1: Identity (13) is a time discrete analogue of the well-known identity for the Schrödinger equation (3) obtained by multipliers, which reads (see Machtyngier, 1994):

$$\frac{1}{2}\int_{\Sigma}(q,\nu)\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Sigma = \frac{1}{2}\operatorname{Im}\int_Q (\varphi q\cdot\nabla\overline{\varphi})\,dx\Big|_0^T + \frac{1}{2}\operatorname{Im}\int_\Omega (q_t\cdot\nabla\varphi\overline{\varphi})dxdt$$
$$+\frac{1}{2}\operatorname{Re}\int_Q(\varphi\nabla(\mathrm{div}_x q)\cdot\nabla\overline{\varphi})dxdt + \operatorname{Re}\int_Q\sum_{i,j}\left(\frac{\partial q_j}{\partial x_i}\frac{\partial\overline{\varphi}}{\partial x_j}\frac{\partial\varphi}{\partial x_i}\right)dxdt. \tag{14}$$

Clearly, the major difference between (14) and (13) is that, φ and φ_t are replaced by $(\varphi_{k+1}+\varphi_k)/2$ and $(\varphi_{k+1}-\varphi_k)/h,$ respectively, due to the time-discretisation. It is easy to see, formally, that $(\varphi_{k+1}+\varphi_k)/2$ and $(\varphi_{k+1}-\varphi_k)/h$ tends to φ and φ_t as $h\to 0,$ respectively. However, this convergence does not hold uniformly for all solutions. This induces the need of using filtering of the high frequencies to obtain observability inequalities, as we shall see in Lemma 4.1.

Proof: The desired identity will be given by using the multiplier

$$q_k\cdot\nabla\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)+\frac{1}{2}\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\mathrm{div}_x q_k. \tag{15}$$

Integrating on Ω and summing k from 1 to $K-1,$ we have

$$h\sum_{k=0}^{K-1}\int_\Omega\left[i\frac{\varphi_{k+1}-\varphi_k}{h}+\Delta\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right]\times(15)dx=0. \tag{16}$$

We denote by

$$A=h\sum_{k=0}^{K-1}\int_\Omega i\frac{\varphi_{k+1}-\varphi_k}{h}q_k\cdot\nabla\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)dx,$$
$$B=h\sum_{k=0}^{K-1}\int_\Omega i\frac{\varphi_{k+1}-\varphi_k}{h}\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{4}\mathrm{div}_x q_k dx,$$
$$C=h\sum_{k=0}^{K-1}\int_\Omega \Delta\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)q_k\cdot\nabla\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)dx,$$
$$D=h\sum_{k=0}^{K-1}\int_\Omega \Delta\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{4}\mathrm{div}_x q_k dx.$$

Using integration by parts with respect to x and recalling that $\varphi_k=0$ on $\partial\Omega,$ it is easy to show that

$$B=\frac{i}{2}h\sum_{k=0}^{K-1}\int_\Omega\nabla\left(\frac{\varphi_{k+1}-\varphi_k}{h}\right)\cdot q_k\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)dx-\frac{A}{2}. \tag{17}$$

On the other side, by deindexing the terms in the sum and with careful arrangement, we have the following identity on A:

$$
\begin{aligned}
\frac{A}{2} &= \frac{i}{4}\sum_{k=0}^{K-1}\int_\Omega \nabla(\varphi_{k+1}+\varphi_k)q_k\cdot\nabla(\overline{\varphi}_{k+1}+\overline{\varphi}_k)dx \\
&= \frac{i}{4}\sum_{k=0}^{K-1}\int_\Omega \big[2(\varphi_{k+1}q_{k+1}\cdot\nabla\overline{\varphi}_{k+1}-\varphi_k q_k\cdot\nabla\overline{\varphi}_k) \\
&\qquad -2\varphi_{k+1}(q_{k+1}-q_k)\cdot\nabla\overline{\varphi}_{k+1} \\
&\qquad -(\varphi_{k+1}+\varphi_k)q_k\cdot\nabla(\overline{\varphi}_{k+1}-\overline{\varphi}_k\big]dx \qquad (18)\\
&= \frac{i}{2}\int_\Omega (\varphi K qK \cdot\nabla\overline{\varphi}K - \varphi 0 q 0\cdot\nabla\overline{\varphi}_0)dx \\
&\quad -\frac{i}{2}h\sum_{k=0}^{K-1}\int_\Omega\bigg(\varphi_{k+1}\frac{q_{k+1}-q_k}{h}\cdot\nabla\overline{\varphi}_{k+1} \\
&\qquad +\frac{\varphi_{k+1}+\varphi_k}{2}q_k\cdot\nabla\frac{\overline{\varphi}_{k+1}-\overline{\varphi}_k}{h}\bigg)dx.
\end{aligned}
$$

Combining (17) and (18), one arrives at

$$
\begin{aligned}
A+B &= \frac{A}{2}-\frac{i}{2}h\sum_{k=0}^{K-1}\int_\Omega \frac{\varphi_{k+1}-\varphi_k}{h}\cdot q_k\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}dx \\
&= \frac{i}{2}\int_\Omega(\varphi K qK\cdot\nabla\overline{\varphi}K-\varphi 0 q 0\cdot\nabla\overline{\varphi}_0)dx \\
&\quad -\frac{i}{2}h\sum_{k=0}^{K-1}\int_\Omega\varphi_{k+1}\frac{q_{k+1}-q_k}{h}\cdot\nabla\overline{\varphi}_{k+1}dx \qquad (19)\\
&\quad -\frac{i}{2}h\sum_{k=0}^{K-1}\int_\Omega\bigg(\frac{\varphi_{k+1}+\varphi_k}{2}q_k\cdot\nabla\frac{\overline{\varphi}_{k+1}-\overline{\varphi}_k}{h} \\
&\qquad +\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}q_k\cdot\nabla\frac{\varphi_{k+1}-\varphi_k}{h}\bigg)dx.
\end{aligned}
$$

On the other side, we similarly compute

$$
\begin{aligned}
C &= h\sum_{k=0}^{K-1}\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 q_k\cdot\nu d\sigma \\
&\quad -h\sum_{k=0}^{K-1}\int_\Omega\sum_{i,j}\left[\frac{\partial q_{j,k}}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)\frac{\partial}{\partial x_i}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right]dxdt \qquad (20)\\
&\quad -h\sum_{k=0}^{K-1}\int_\Omega\nabla\frac{\varphi_{k+1}+\varphi_k}{2}\cdot q_k\Delta\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}dx.
\end{aligned}
$$

Moreover,

$$D = -\frac{1}{2}h\sum_{k=0}^{K-1}\int_{\Omega}\nabla\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\cdot\nabla\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)\text{div}_x q_k dx$$
$$-\frac{1}{2}h\sum_{k=0}^{K-1}\int_{\Omega}\nabla\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\cdot\left[\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)\nabla\text{div}_x q_k\right]dx. \tag{21}$$

Using integration by parts on x, after some computations, (21) changes to

$$D = -\frac{1}{2}h\sum_{k=0}^{K-1}\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 q_k\cdot\nu d\sigma + \frac{C}{2}+\frac{\overline{C}}{2}$$
$$-\frac{1}{2}h\sum_{k=0}^{K-1}\int_{\Omega}\nabla\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\cdot\left[\left(\frac{\overline{\varphi}_{k+1}+\overline{\varphi}_k}{2}\right)\nabla\text{div}_x q_k\right]dx. \tag{22}$$

Combining (19)–(22), taking their real parts, we arrive at (13).

4 Uniform observability under filtering

In this section, we will introduce the filtering method to get rid of the high frequencies involved in the propagation of the time discrete model. Follow this, we show a positive result claiming that (5) is exact observable uniformly on h.

To begin with, we introduce the following filtering space in which the solutions involved in:

$$C_s = \{g(x) \mid g(x) = \sum_{\mu_j^2 < s} b_j \Phi j(x),\ b_j \in \mathbb{C}\} \subset H_0^1(\Omega)\},$$

where μ_j^2 and Φ_j are defined in (7). We claim that the solution of system (5) can be expressed by means of Fourier series. Indeed, we have

Lemma 4.1: Assume $\varphi_0 = \sum_{j=1}^{\infty} a_j\Phi_j$. Then the corresponding solution of (5) with initial data φ_0 has the form

$$\varphi_k = \sum_{j=1}^{\infty} a_j \exp(-i\lambda_j kh)\Phi_j, \quad \textit{with} \quad \lambda_j = \frac{2}{h}\arctan\left(\frac{u_j^2 h}{2}\right). \tag{23}$$

Moreover, $\varphi_0 = \sum_{\mu_j^2 \le s}^{\infty} a_j\Phi_j \in \mathcal{C}_s$, it holds

$$\int_{\Omega}\left|\nabla\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 dx \ge \frac{4}{4+(sh)^2}\|\varphi_0\|_{H_0^1(\Omega)}^2. \tag{24}$$

Remark 4.1: Note that (24) induces the major difference when one considers the observability rather than other classical ones, i.e., convergence of the solution, stability, etc. More precisely, most of the classical properties concern one single solution, but observability concerns a 'uniform' inequality for a class of solutions. For any specified solution, the left hand side of (24) is an approximation of the right hand side, as time step h tends to zero. However, when $\varphi_0 \in H_0^1(\Omega)$, (24) shows that it will no longer be true when the solutions containing more and more high frequencies. The critical case is arrived if sh is a constant, where s is the largest eigenvalues containing in the system. It is also the crucial point for testifying the optimal order of the filtering parameter, as we will see later.

Proof: It can be done by simple computations.

Now we establish uniform observability estimates for system (5) (with respect to time h) after filtering the spurious high frequency components:

Theorem 4.1: Let Γ_0 satisfy (1). Let $T > 0$ and K be a positive integer. Let $h = T / K$ and $\alpha = \min(2/d, 1)$. For any $\delta > 0$, there exist $T_\delta, C_1, C_2 > 0$, independent of h, such that for the solutions of (5),

$$h\sum_{k=0}^{K-1}\int_{\Gamma}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 d\Gamma \le C_1 \|\varphi_0\|^2_{H_0^1(\Omega)} \tag{25}$$

and

$$\|\varphi_0\|^2_{H_0^1(\Omega)} \le C_2 h\sum_{k=0}^{K-1}\int_{\Gamma_0}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 d\Gamma_0 \tag{26}$$

hold for all $T > T_\delta$, $h > 0$ and $\varphi_0 \in \mathcal{C}_{\delta/h^\alpha}$.

Remark 4.2: Note that (26) is a time discrete version of the continuous observability estimate. Consequently, since C_2 is a constant independent of h and $\mathcal{C}_{\delta/h^\alpha}$ tends to $H_0^1(\Omega)$ as h tends to zero, Theorem 4.1 can recover the observability property (4) for the continuous system (3).

Remark 4.3: Note that one can deduce that uniform observability holds for any time $T > 0$. In fact, the relations between T_δ and δ (see (28) and (47)) tells us that T_δ tends to zero as δ vanishes. Hence, for any $T > 0$, there exists a sufficiently small δ such that the uniform observability (26) holds for any $\varphi_0 \in \mathcal{C}_{\delta/h^\alpha}$.

Remark 4.4: Note that the order of the filtering parameter is optimal as $d = 1, 2$. Recall that $\alpha = \min(2/d, 1)$, we have $\alpha = 1$ as $d = 1, 2$ and $\alpha = 2/d$ as $d > 2$. On the other hand, as we will see later, Theorem 5.1 will show that the counterexample appears when α is bigger than 1. Combining these two facts we can arrive the optimal filtering order 1 as $d = 1, 2$. Due to the technical limitation, it is still an open problem whether $\alpha = 2/d$ is optimal or not.

Proof: We first prove (25). We choose $q_k = q(x) = C^2(\overline{\Omega}, \mathbb{R}^d)$ such that $q = \nu$ on Γ (see Lions, 1988 for the construction of this vector field), and we obtain

$$\begin{aligned}
&\frac{1}{2}h\sum_{k=0}^{K-1}\int_\Gamma \left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 q\cdot\nu d\Gamma \\
&\le k_1\|q\|_{L\infty(\Omega)}\left(\|\varphi_K\|^2_{L^2(\Omega)}+\|\nabla\varphi_K\|^2_{L^2(\Omega)}+\|\varphi_0\|^2_{L^2(\Omega)}+\|\nabla\varphi_0\|^2_{L^2(\Omega)}\right) \\
&+k_2\|q\|_{W^{2,\infty}(\Omega)}\,h\sum_{k=0}^{K-1}\left\|\frac{\varphi_{k+1}+\varphi_k}{2}\right\|_{L^2(\Omega)}\left\|\nabla\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right\|_{L^2(\Omega)} \\
&+k_3\|q\|_{W^{1,\infty}(\Omega)}\,h\sum_{k=0}^{K-1}\left\|\nabla\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right\|^2_{L^2(\Omega)}.
\end{aligned} \tag{27}$$

As we shall see in Lemma 6.1, the L^2-norms of φ_k and $\nabla\varphi_k$ are conserved for any k. Taking (45) into account, for any $h > 0$, there exists a constant $C_1 > 0$, independent of h, such that

$$h\sum_{k=0}^{K-1}\int_\Gamma \left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 d\Gamma \le C_1\|\varphi_0\|^2_{H_0^1(\Omega)}, \quad \forall\varphi_0\in\mathcal{D}(\Omega).$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, the estimate (25) holds for every solution of (3.1) with initial data $\varphi_0 \in H_0^1(\Omega)$.

Now we prove (26).

Step 1 We first prove the following inequality:

$$\left(\frac{4T}{4+\delta^2}-\varepsilon\right)\|\varphi_0\|^2_{H_0^1(\Omega)} \le \frac{h}{2}\sum_{k=0}^{K-1}\int_{\Gamma_0}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 d\Gamma_0 + C_\varepsilon\|\varphi_0\|^2_{L^2(\Omega)}. \tag{28}$$

Recalling the identity (13), setting $q_k = m(x) = x - x_0$ and $f_k = 0$ for any k, we get

$$\begin{aligned}
&\frac{1}{2}h\sum_{k=0}^{K-1}\int_\Gamma \left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 m\cdot\nu d\Gamma \\
&=\frac{1}{2}\mathrm{Im}\int_\Omega(\varphi_K m\cdot\nabla\overline{\varphi}_K-\varphi_0 m\cdot\nabla\overline{\varphi}_0)dx + h\sum_{k-0}^{K-1}\int_\Omega\left|\nabla\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 dx.
\end{aligned} \tag{29}$$

Furthermore, let $\varepsilon > 0$ sufficiently small, we get

$$\frac{1}{2}\mathrm{Im}\int_\Omega(\varphi_K m\cdot\nabla\overline{\varphi}_K-\varphi_0 m\cdot\nabla\overline{\varphi}_0)dx + |\le C\varepsilon\|\varphi_0\|^2_{L^2(\Omega)}+\varepsilon\|\varphi_0\|^2_{H_0^1(\Omega)}. \tag{30}$$

Thus, recalling Lemma 4.1, combining (29) and (30), we arrive at (28). Note that ε has to be chosen such that the constant on the left hand side of (28) is positive, i.e., $\varepsilon < \dfrac{4T}{4+\delta^2}$.

Step 2 We now prove the following estimate, which plays the key role for reducing (28) to our desired inequality (26): There exists $K > 0$, independent of h and φ_0 such that

$$\|\varphi_0\|_{L^2(\Omega)} \le Kh\sum_{k=0}^{K-1}\int_{\Gamma_0}\left|\frac{\partial\varphi_k}{\partial\nu}\right|^2 d\Gamma_0. \tag{31}$$

We argue by contradiction.

First, we prove that K is not depending on φ_0. For an $\varphi_0 \in \mathcal{C}_{\delta/h^\alpha}$, if (31) is not satisfied for any $K > 0$, there exists a sequence $\{\varphi_0^n\}$ of initial data of (5) such that

$$\|\varphi_0^n\|_{\mathcal{C}_{\delta/h^\alpha}} = 1, \text{ i.e., } \|\varphi_0^n\|_{L^2(\Omega)} = 1 \quad \forall n \in \mathbb{N} \tag{32}$$

and

$$h\sum_{k=0}^{K-1}\int_{\Gamma_0}\left|\frac{\partial\varphi_k}{\partial\nu}\right|^2 d\Gamma_0 \to 0 \quad \text{as} \quad n \to +\infty. \tag{33}$$

Note that in (33) $\{\varphi_k^n\}$ denotes the corresponding solutions of (5) with initial data φ_0^n.

Since $\mathcal{C}_{\delta/h^\alpha}$ is a finite dimensional subspace of $H_0^1(\Omega)$, from (28) we deduce that $\{\varphi_0^n\}$ is bounded in $H_0^1(\Omega)$. By extracting a subsequence (that we will still note by $\{\varphi_0^n\}$) we will have

$$\{\varphi_0^n\} \to \varphi_0 \text{ weakly in } H_0^1(\Omega),$$

which implies

$$\{\varphi_0^n\} \to \varphi_0 \text{ strongly in } L^2(\Omega).$$

From (32) we deduce

$$\|\varphi_0\|_{L^2(\Omega)} = 1. \tag{34}$$

On the other hand, (33) implies

$$\frac{\partial\varphi_k^n}{\partial\nu} = 0 \text{ on } \Gamma_0.$$

Since $\varphi_k \in \mathcal{C}_{\delta/h^\alpha}$, from Lemma 6.2 we get $\varphi_k \equiv 0$. This is in contradiction with (34). This means that K in (31) is independent of φ_0.

Using the same argument as above, it is easy to show that K is also independent of h.

Indeed, the above argument holds true for any $h > 0$. More precisely, if K blows up when h tends to zero, we have

$$\lim_{h\to 0}\sup_{\varphi_0\in\mathcal{C}_{\delta/h^\alpha}}\frac{\|\varphi_0\|^2_{L^2(\Omega)}}{h\sum_{k=0}^{K-1}\int_{\Gamma_0}\left|\frac{\partial\varphi_k}{\partial\nu}\right|^2 d\Gamma_0}=\infty. \tag{35}$$

Consequently, for any $K>0$, there exists $\varphi_0\in\mathcal{C}_{\delta/h^\alpha}\cap L^2(\Omega)$ and $h>0$ such that

$$h\sum_{k=0}^{K-1}\int_{\Gamma^0}\left|\frac{\partial\varphi k}{\partial\nu}\right|^2 d\Gamma_0\le\varepsilon,$$

where $\varepsilon\to 0$ as $K\to\infty$. Hence, we can choose a sequence $\left\{\varphi_0^n\right\}$ of the initial data of (5) such that it satisfies (32) and (33). Using the same argument above, remind that (28) holds for any $h>0$, the same contradiction comes out. This means that K is independent of h too.

Combining these two facts the proof of inequality (31) is complete.

Note that compared to the continuous level, it is necessary to show that K is a constant independent of both the initial state φ_0 and the time step h. The contradiction method here still works due to the fact that (28) holds uniformly not only for any $\varphi_0\in\mathcal{C}_{\delta/h^\alpha}\cap L^2(\Omega)$ but for any $h>0$ too, which is provided by the appropriate filtering technique.

Step 3 Now we derive (26) by means of (28) and (31). Assume $\varphi_0=\sum_{\mu_j^2\le\delta/h^\alpha}a_j\Phi_j$.

Let

$$\psi_0=\sum_{\mu_j^2\le\delta/h^\alpha}a_j\exp(-i\lambda_j h/2)\cos(\lambda_j h/2)\Phi_j.$$

It is obvious that 0 $\psi_0\in\mathcal{C}_{\delta/h^\alpha}$. Consequently, (28) holds by replacing φ_k by ψ_k.

Meanwhile, using (23), it is easy to show that its corresponding solution has the form

$$\psi_k=\sum_{\mu_j^2\le\delta/h^\alpha}a_j\exp(-i\lambda_j h/2)\cos(\lambda_j h/2)\exp(-i\lambda_j kh)\Phi_j=\frac{\varphi_{k+1}+\varphi_k}{2}.$$

Moreover, taking (31) into account, it holds

$$\begin{aligned}\|\psi_0\|^2_{L^2(\Omega)}&\le Kh\sum_{k=0}^{K-1}\int_{\Gamma_0}\left|\frac{\partial\psi_k}{\partial\nu}\right|^2 d\Gamma_0\\&=Kh\sum_{k=0}^{K-1}\int_{\Gamma_0}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 d\Gamma_0.\end{aligned} \tag{36}$$

On the other side, since $\mu_j^2\le\delta/h$ and recalling (23), we compute

$$\begin{aligned}\|\psi_0\|^2_{L^2(\Omega)} &= \sum_{\mu_j^2\le\delta/h^\alpha} |a_j|^2 |\cos(\lambda_j h/2)|^2 \\ &\ge \frac{4}{4+\delta^2}\sum_{\mu_j^2\le\delta/h^\alpha} |a_j|^2 = \frac{4}{4+\delta^2}\|\varphi_0\|^2_{L^2(\Omega)}.\end{aligned} \tag{37}$$

Due to the fact that ψ_k satisfies (28), combining (36) and (37), we conclude (26).

5 Optimality of the order of filtering parameter

In this section we will discuss the optimality of the filtering mechanism introduced in Theorem 4.1. We have the following Theorem:

Theorem 5.1: Assume that Γ_* is any nonempty open set of Γ. Then for any given $\beta > 1$, it follows that

$$\lim_{h\to 0}\sup_{\varphi_0^h\in\mathcal{C}_{h^{-\beta}}} \frac{\|\varphi_0^h\|^2_{H_0^1(\Omega)}}{h\sum_{k=0}^{K-1}\int_{\Gamma_*}\left|\frac{\partial}{\partial\nu}\left(\frac{\varphi_{k+1}+\varphi_k}{2}\right)\right|^2 d\Gamma_*} = \infty. \tag{38}$$

Remark 5.1: Note that $h^{-\alpha}$, the order of filtering parameter h, is optimal when $d < 3$. In fact, $\alpha = \min(2/d, 1) = 1$ as $d < 3$. However, it is still an open problem whether it is sharp in the case $d \ge 3$. One of the possibility to solve this problem is to establish a time-discrete Carleman estimate for the time-discrete system (5) and show its corresponding unique continuation property.

Proof: Recall that $\{\Phi_j\}_{j=1}^\infty \subset H_0^1(\Omega)$ denotes the orthonormal basis of $L^2(\Omega)$ constituted by the eigenvectors of the Dirichlet Laplacian and $\left\{\mu_j^2\right\}_{j\ge 1}$ the corresponding eigenvalues. Since $\mu_j^2 \to +\infty$ as $j\to\infty$, one can choose a $j_0 = j_0(h)$ so that $h^{-\beta}/2 \le \mu_{j_0}^2 \le h^{-\beta}$. In view of the fact that $\beta > 1$, this leads to

$$\mu_{j_0}^2 h \to \infty, \text{ as } h \to 0. \tag{39}$$

Further, choose

$$\varphi_0^h = \frac{1}{\mu_{j0}}\Phi_{j0}. \tag{40}$$

One deduces that $\varphi_0^h \in \mathcal{C}_{h^{-\beta}}$ and $\left\|\varphi_0^h\right\|_{H_0^1(\Omega)} = 1$. Noting the special choice of initial data in (40), by Lemma 4.1, the corresponding solution $\left\{\varphi_k\right\}_{k=0,\dots,K}$ of (5) is given by

$$\varphi k = \frac{1}{\mu_{j0}} \exp(-i\lambda_{j0} kh)\Phi_{j0}, \quad k = 0,...,K, \tag{41}$$

where λ_{j0} is defined by (23). Using (23), it follows

$$\cos\left(\frac{\lambda_{j0}h}{2}\right) = \frac{2}{\sqrt{4 + (\mu_{j0}^2 h)^2}}. \tag{42}$$

Via (41) and (42), one has

$$\begin{aligned} &\int_{\Gamma_*} \left| \frac{\partial}{\partial \nu}\left(\frac{\varphi^{k+1} + \varphi^k}{2}\right)\right|^2 d\Gamma_* \\ &\le \left| \frac{e^{i\lambda_{j0}h/2} + e^{-i\lambda_{j0}h/2}}{2\mu_{j0}} \right|^2 \int_{\Gamma_*} \left| \frac{\partial \Phi_{j0}}{\partial \nu} \right|^2 d\Gamma_* \le \frac{\cos^2(\lambda_{j0}h/2)}{\mu_{j0}^2} \int_{\Gamma} \left| \frac{\partial \Phi_{j0}}{\partial \nu} \right|^2 d\Gamma. \end{aligned} \tag{43}$$

From (7.8) of Zhang et al. (2009) we know that

$$\int_{\Gamma} \left| \frac{\partial \Phi_{j0}}{\partial \nu} \right|^2 d\Gamma \le C\mu_{j0}^2. \tag{44}$$

Finally, combining (43) and (44), noting (39), (42) and $\left\| \varphi_0^h \right\|_{H_0^1(\Omega)} = 1$, it follows

$$\sup_{\varphi_0^h \in \mathcal{C}_{h^{-\beta}}} \frac{\left\| \varphi_0^h \right\|_{H_0^1(\Omega)}^2}{h \sum_{k=1}^{K-1} \int_{\Gamma_*} \left| \frac{\partial}{\partial \nu}\left(\frac{\varphi^{k+1} + \varphi^k}{2}\right)\right|^2 d\Gamma_*} \ge \frac{4 + \left(\mu_{j0}^2 h\right)^2}{4C} \to \infty \text{ as } h \to 0,$$

which gives (38).

6 Further comments and open problems

1 In this paper we dealt with the time discrete implicit-midpoint schemes (5). In fact, uniform observability holds true for any norm-conserved schemes under suitable filtering conditions of the initial data (see Ervedoza et al., 2008), such as Newmark method, Gauss method, etc. One could expect the same relationship between the filtering parameter and the optimal observable time T (it is not clear in Ervedoza et al., 2008). It is the key improvement of this paper. i.e., recovers the optimal time in the continuous level. It is interesting to do further works as an exercise on these much more complicated systems. The main heavy work is the complexity of the computations on multiplier techniques acting on much more complicated time discrete schemes.

2 As it is well-known in controllability theory, uniform observability inequalities imply uniform controllability results as well. For instance, similarly as the time discrete wave equation analysed in Zhang et al. (2009), combining the duality arguments and the results of this paper, one can immediately deduce the uniform (with respect to $h > 0$) controllability of projections on the class of filtered space $\mathcal{C}_{\delta/h^\alpha}$ for arbitrary $T > 0$.

3 Another interesting further open problem is whether the fully discrete schemes have the uniform observability properties. For instance, in one dimension, one can replaces the continuous operator d_{xx} by the central schemes $(\mu_{j+1} + \mu_{j-1} - 2u_j)/(\Delta x)^2$, where μ_j indicates the approximations of u at $x = j\Delta x$. Obviously, the complexity increases as the dimension increases. We will discuss this problem elsewhere.

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Appendix Some Lemmas

Lemma 6.1: The solutions of the time discrete Schrödinger equation (5) satisfy

$$\begin{aligned}&\|\varphi_k\|^2_{L^2(\Omega)} = \|\varphi_0\|^2_{L^2(\Omega)}, \quad \forall k = 0,\dots,K,\\ &\|\nabla\varphi_k\|^2_{L^2(\Omega)} = \|\nabla\varphi_0\|^2_{L^2(\Omega)}, \quad \forall k = 0,\dots,K.\end{aligned} \tag{45}$$

Proof: Identity (45), which is an analogue of the classic results of the Schrödinger equation, can be proved directly by using the following formulas:

$$h\sum_{k=0}^{K-1}\int_\Omega\left((5)\times\overline{\frac{\varphi_{k+1}+\varphi_k}{2}} - \overline{(5)}\times\frac{\varphi_{k+1}+\varphi_k}{2}\right)dx = 0,$$
$$h\sum_{k=0}^{K-1}\int_\Omega\left((5)\times\overline{\frac{\varphi_{k+1}-\varphi_k}{h}} - \overline{(5)}\times\frac{\varphi_{k+1}-\varphi_k}{h}\right)dx = 0.$$

Lemma 6.2: Let $T > 0$, $K = T/h$. Then there exists $\delta = \delta(\Omega, T) > 0$, such that for any $\varphi_0 \in \mathcal{C}_{\delta/h^\alpha}$ with $\alpha = \min(2/d, 1)$, the solution of (5) has the property

$$\frac{\partial\varphi_k}{\partial\nu} = 0, \text{ on } \Gamma_0, \ \forall k = 0,\dots,K, \ \Rightarrow \ \varphi_k \equiv 0, \text{ in } \Omega, \ \forall k = 0,\dots,K.$$

Remark 6.1: Note that Lemma 6.2 is a partial unique continuation property of system (5). The existence of a counterexample seems to be easily given by taking a nontrivial solution of system (46). However, the explicit forms of eigenvalues and eigenfunctions are hard to be caught in general case, saying, for an optional Ω. Whether there exists an general datum violating the unique continuation property is still unknown.

Proof: By Weyl's formula $\mu_j^2 \sim C(\Omega) j^{2/d}$, For any $\delta > 0$ there exists an integer J such that the initial data $\varphi_0 \in \mathcal{C}_{\delta/h^\alpha}$ can be written as the form

$$\varphi_0 \sum_{j=1}^{J}\sum_{l=1}^{n_j} a_{j,l}\Phi_{j,l},$$

where $\Phi_{j,l}, l = 1,\dots,n_j$ are eigenfunctions corresponding to the same eigenvalue λ_l. By Lemma 4.1, the solution of (5) is

$$\varphi_k = \sum_{j=1}^{J}\sum_{l=1}^{n_j} a_{j,l}\exp(-i\lambda_j kh)\Phi_{j,l}.$$

Taking $\frac{\partial\varphi_k}{\partial\nu} = 0$ into account, it holds

$$\begin{cases} \displaystyle\sum_{j=1}^{J}\sum_{l=1}^{n_j} a_{j,l}\partial_\nu \Phi_{j,l} = 0, \\ \displaystyle\sum_{j=1}^{J}\exp(-i\lambda_j h)\sum_{l=1}^{n_j} a_{j,l}\partial_\nu \Phi_{j,l} = 0, \\ \qquad\qquad\cdots\cdots \\ \displaystyle\sum_{j=1}^{J}\exp(-i\lambda_j Kh)\sum_{l=1}^{n_j} a_{j,l}\partial_\nu \Phi_{j,l} = 0. \end{cases} \tag{46}$$

Or, equivalently,

$$MA = 0,$$

with

$$M = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \exp(-i\lambda_1 h) & \exp(-i\lambda_2 h) & \dots & \exp(-i\lambda_J h) \\ \dots & \dots & \dots & \dots \\ \exp(-i\lambda_1 Kh) & \exp(-i\lambda_2 Kh) & \dots & \exp(-i\lambda_J Kh) \end{bmatrix} \triangleq \begin{bmatrix} M_0 \\ M_1 \\ \dots \\ M_K \end{bmatrix},$$

and

$$A = \begin{bmatrix} \displaystyle\sum_{l=1}^{n_1} a_{1,l}\partial_\nu \Phi_{1,l} \\ \displaystyle\sum_{l=1}^{n_2} a_{2,l}\partial_\nu \Phi_{2,l} \\ \dots\dots \\ \displaystyle\sum_{l=1}^{n_J} a_{J,l}\partial_\nu \Phi_{J,l} \end{bmatrix} \triangleq \begin{bmatrix} A_1 \\ A_2 \\ \dots \\ A_J \end{bmatrix}.$$

Obviously, $A_j \equiv 0$ if and only if $J \le K+1$ since M is a $(K+1)\times J$ Van der Monde matrix. More precisely, recalling that $\lambda_j = \frac{2}{h}\arctan\left(\frac{\mu_j^2 h}{2}\right)$ in (23), it is easy to verify that $\lambda_i \ne \lambda_j$ when $i \ne j$, and consequently $\exp(i\lambda_i h) \ne \exp(i\lambda_j h)$ (due to the fact that $\lambda_j h \in (-\pi, \pi)$ for any j). If $J < K+1$, the first J equations of (46) indicates that $A_j \equiv 0$; If $J = K+1$, $\det(M) = 0$ and $MA = 0$ only has trivial solution, i.e., $A_j \equiv 0$. Combining these two facts we arrive at $J \le K+1$.

Meanwhile, to provide $J \le K+1$, it is sufficient that $\alpha = \min(2/d, 1)$ and to make an appropriate choice of δ. In fact, since $\mu_J^2 \sim C(\Omega) J^{2/d} \le \delta / h^\alpha$, to let $J \le K+1$, it is sufficient to choose δ such that the following inequality is fulfilled:

$$\frac{\delta}{h^{\alpha}} \le C(\Omega)(K+1)^{2/d} = C(\Omega)\left(\frac{T}{h}+1\right)^{2/d} \tag{47}$$

due to the fact that

$$\left.\begin{aligned} &\frac{\delta}{h^{\alpha}} \le C(\Omega)(K+1)^{2/d} \\ &\mu_J^2 \sim C(\Omega)J^{2/d} \le \frac{\delta}{h^{\alpha}} \end{aligned}\right\} \Rightarrow J \le K+1. \tag{48}$$

Clearly, there always exists a positive constant δ independent of $h,$ such that (47) holds by the assumption $\alpha = \min(2/d, 1).$

On the other hand, since $\Phi_{j,l}$ satisfies $\Delta\Phi_{j,l} = -\mu_J^2\Phi_{j,l}$ and $\Phi_{i,}\Phi_{j,}$ are linear independent when $i \ne j,$ it holds

$$A_j = \sum_{l=1}^{n_j} a_{j,l}\partial_\nu\Phi_{j,l} = 0 \;\Rightarrow\; a_{j,l} = 0,\;\forall\; l \in [1, n_j].$$

Combining this fact and that $A_j \equiv 0$ for $j = 1, ..., J,$ we complete the proof.